The Riemann Hypothesis

Chapter 1

Historical Background: Cash Money and Other Mathematical Concepts

1.1 He's Making a List, and Checking it for Money

On August 8, 1900, David Hilbert, a German mathematician with a penchant for really cool-looking hats, strode to the board at the International Congress of Mathematicians in Paris and, in an historic pronouncement, unveiled a list of ten problems that he felt would be of fundamental importance to mathematics in the 20th century. The mathematicians around the room were shocked, presumably because many of them had never seen ten problems listed in a row before[1]. Critical acclaim for the pronouncement was immediate, with some critics going so far as to call it "The talk of the new century!" and "The best list of ten problems I've ever seen!"

Encouraged by his success[2], Hilbert soon released the notes from his talk with a director's commentary in which he announced thirteen extra problems, making twenty-three in total. These problems, like their predecessors, were also met with great approval from the mathematical community.

It often happens that a work, though critically acclaimed, is quickly for-

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[1]Plus, several of them had been studying mathematical abstractions for so long that they had forgotten how to count to ten, making Hilbert's enumeration even more shocking.

[2]Although disappointed that no one had commented on how awesome his hat looked.

gotten. This was not the case with Hilbert's problems. In fact, the result was quite the opposite, as Hilbert's list turned out to be creepily predictive of the new century. Today, seventeen of the twenty-three questions have been fully or partially resolved; moreover, for many of these questions, the solutions (or partial solutions) resulted in breakthroughs that became fundamental to the development of 20th century mathematics. Just six of the questions remain completely unsettled, although one of the questions ("Construct all metrics where lines are geodesics") is considered hopelessly broad and another question ("Is this hat awesome or what?") has been deemed by many mathematicians to be "too rhetorical to pursue."

Exactly one hundred years later, with most of Hilbert's original list solved, the Clay Mathematics Institute decided to honor Hilbert's announcement by compiling a new list of of twenty-three unsolved problems that would be of importance in the 21st century. Unfortunately, coming up with a large list of interesting unsolved problems turns out to be harder than it looks, and the Institute managed to compile just seven problems (including one - the Riemann Hypothesis - that they "borrowed" from Hilbert's original list) before they were forced to resort to rhetorical questions about hats. Realizing that a list of just seven problems, one of which wasn't even new, wouldn't exactly electrify the math community, the Clay Mathematics Institute decided that the best way to garner attention and support for their list was to add some "pizazz" (defined here as "large quantities of money") to it, and so they announced that anyone who solved any of the seven problems on the list would receive a one million-dollar prize. The gambit worked; mathematicians, who were now able to claim with a straight face that they worked in a "lucrative field" when talking to friends or inquisitive in-laws, gave the Institute's announcement a warm reception, and the list soon came to be accepted as the natural successor to Hilbert's list.

Of all of the mathematical concepts on the Clay Mathematics list, the most interesting one for a number theorist is undoubtedly the one million dollars. However, that doesn't make for a very interesting paper, so we'll talk instead about the second most interesting one: the aforementioned Riemann Hypothesis.

Much to the amazement of his colleagues, David Hilbert could count all the way up to twenty-three.

1.2 The Riemann Hypothesis: Yeah, I'm Jealous

The Riemann Hypothesis is named after the fact that it is a hypothesis, which, as we all know, is the largest of the three sides of a right triangle. Or maybe that's "hypotenuse." Whatever. The hypothesis was posed in 1859 by Bernhard Riemann, a mathematician who was not a number theorist and wrote just one paper on number theory in his entire career. Naturally, this single paper would go on to become one of the most influential papers in number theory history, a depressing, frustrating, and angering thought for those of us who will actually work in number theory full-time for our entire lives and will still never write a number theory paper nearly that important.

In this infuriatingly important paper, entitled *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* (which, according to a translator I found online, translates into English as *Possibility of the Size Lower Part Primes Which Comes to Give*), Riemann introduced mysterious new function which he called ζ (the Greek letter zeta) and asked, "When does this function hit zero?" While this may seem a really simple question, the zeta function turns out to be extremely important to the study of number theory, and Riemann's query has gone on to have far-reaching implications not just in number theory but throughout all of math and physics. What a jerk.

Of course, Riemann's work was predicated upon the earlier work of Leonard Euler. As you will soon discover, this is actually one of the recurring literary themes in this book; most of the stuff that we do in number theory is based on some idea that Euler had, so his name comes up an awful lot. Apparently, Euler was a pretty productive guy.

In this case, what Euler did was discover a new function that would give rise to this "zeta" that I mentioned above. We'll outline Euler's function and the sorts of bizarre things Riemann did to it in the next chapter.

Chapter 2

Make My Funk a Z-Func(tion)

To help us understand the function that Euler discovered, let us begin with a simple question. What happens if we take the following sum:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + ...?$$

What happens is that the sum goes off to infinity, indicating that the above is a dumb question.

Lets try again. What happens if we take another sum:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + ...?$$

This one is actually a little more interesting:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + ... = \frac{\pi^2}{6}.$$

The sum

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + ...$$

is a bit of a mystery that I won't get into here (though it equals about 1.202), but this one's kind of cool:

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + ... = \frac{\pi^4}{90}$$

Euler saw all of these identities[1] and was so inspired that he asked the following question:

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[1] Actually, he was the one who discovered them.

The Euler Question: *Get a load of this expression*:

$$\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + ...$$

Are there any s's for which interesting things happen (besides 2 and 4)?

When someone asks a good question like this, the first thing that we mathematicians do is start naming everything in the problem because naming things is way easier than solving math problems. Let's give our expression a name:

Definition: *Write the above expression as* $Z(s)$, *i.e.*

$$Z(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + ...$$

So $Z(2) = \frac{\pi^2}{6}$, $Z(4) = \frac{\pi^4}{90}$, $Z(1)$ goes off to infinity, etc.

Now, we can restate the question above, which still doesn't change anything but again gives us the illusion of doing something productive:

The More Economical Euler Question: *What happens with* $Z(s)$ *when s isn't 2 or 4?*

It looks like we've succeeded in asking our question in as few characters as possible. That's progress.

Unfortunately, the answer isn't going to be quite as nice as we hoped.

Partial Answer: *There are many, many values that are completely uninteresting.*

What's the problem? Well, remember how we said that $Z(1)$ went off to infinity? It turns out that "goes off to infinity" is not an interesting thing for $Z(s)$ to do[2]. Moreover, it turns out that 1 isn't the only place where s goes off to infinity. For example, if you take any positive integer and raise it

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[2]Okay, I suppose it's a little interesting, but compared to something like $\frac{\pi^2}{6}$? Come on. It's no contest.

to the zero, you get 1. So

$$Z(0) = \frac{1}{1^0} + \frac{1}{2^0} + \frac{1}{3^0} + ... = \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + ... = 1 + 1 + 1 + ...$$

which undoubtedly goes off to infinity as well.

It gets even worse if s is a negative number. Remember that having a negative exponent flips the fraction over (i.e. $\frac{1}{x^{-2}} = x^2$). So for things like $s = -1$, we have

$$Z(-1) = \frac{1}{1^{-1}} + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + ... = 1^1 + 2^1 + 3^1 + ...$$

which is also getting really, really big. This is obviously going to be a problem for any negative s. Basically, we have a function that's not even going to be defined half of the time.

From these observations, we have the following theorem to describe just how annoying and useless $Z(s)$ can be:

Major Theorem: *If $s \leq 1$ then the function $Z(s)$ blows.*[3]

2.1 Why Was Euler Thinking About That Function, Anyway?

It's a funny story, actually. One day, the positive integers were minding their own business, patiently waiting in line to be added up in the Z-function, hoping that the operator hadn't chosen an s that would make them all blow up, when all of a sudden.....

....a rebel gang of numbers showed up:

$$\left(\frac{1}{1 - \frac{1}{2^s}} \right) \left(\frac{1}{1 - \frac{1}{3^s}} \right) \left(\frac{1}{1 - \frac{1}{5^s}} \right) \left(\frac{1}{1 - \frac{1}{7^s}} \right) \left(\frac{1}{1 - \frac{1}{11^s}} \right) \left(\frac{1}{1 - \frac{1}{13^s}} \right) ....$$

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[3]Here, by "blows," we of course mean the conventional mathematical definition of "blows up to infinity." I don't know why you would have thought I meant something else.

Good God, thought the integers, those are the prime numbers! And they've arranged themselves in a pattern to make themselves equal to $Z(s)$!

$$Z(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s}....$$

$$= \left(\frac{1}{1 - \frac{1}{2^s}}\right) \left(\frac{1}{1 - \frac{1}{3^s}}\right) \left(\frac{1}{1 - \frac{1}{5^s}}\right) \left(\frac{1}{1 - \frac{1}{7^s}}\right) \left(\frac{1}{1 - \frac{1}{11^s}}\right) \left(\frac{1}{1 - \frac{1}{13^s}}\right)....$$

The positive integers were aghast at the insolence of those primes. The primes had found a vehicle with which they could turn statements about integers into statements about prime numbers. And they had done it by hijacking the integers' beloved $Z(s)$, too!

While the non-prime integers may still harbor some resentment over this coup, we number theorists view this revolution in much the same way Americans view the American Revolution. It means that if we want to think about questions related to prime numbers, we can use this $Z(s)$ to translate them to questions about regular old positive integers, which are much easier to deal with[4]. Since number theorists are obsessed with primes, this is, in the eloquent words of one high-ranking federal official, a "big f***ing deal."

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[4]Don't believe me that positive integers are easier than primes to deal with? Okay, smarty pants, answer me these questions: What's the next prime number after 7549? What's the next positive integer after 7549? Which question was easier? Yeah, that's what I thought.

Chapter 3

The Zeta Function: Magical, Mystical, and....Dear God, What Is That Thing?

Despite the fundamental importance of $Z(s)$, Euler's efforts to tame this function and domesticate it and maybe make it do tricks were hamstrung by the fact that it always seemed to be blowing up at inopportune times. As a result, he gave up and spent the remainder of his life going blind working on other mathematics in a dark attic.

The function continued to lay prostrate in its useless state for over half of a century until Bernhard Riemann came along. He took the function into his office, watched it self-destructively blow up any time a negative number was mentioned, and decided that it needed help. Although Riemann was not a number theorist by trade, he felt, much like that guy in the movie "Lorenzo's Oil," that he could teach himself enough number theory to cure Z of its horrible ailment.

Bertrand Riemann: Hypothesizin'

After months in his basement laboratory[1], Riemann emerged with what he thought was a remedy: a new function that he called ζ (the Greek letter zeta) because Riemann mistakenly thought that he was Greek. It was a function that had the scientific importance of the Frankenstein monster[2] and the aesthetic appeal of, well, the Frankenstein monster. Take a look at this thing:

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[1]Riemann probably didn't have a basement laboratory. He probably had a cushy university office with a comfy couch where he would lie down and sometimes take naps, and then sometimes people would knock on his office door and he would groggily tell them that he wasn't napping but instead "thinking about mathematics." Believe me, that little trick isn't fooling anybody, Riemann.

[2]I suppose that the Frankenstein monster didn't have any actual scientific importance since it was a fictional monster, but the scientists in the movie all looked pretty impressed, so that's close enough for me.

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!}(k+1)^{-s}.$$

It's hideous[3]!

Why is this a fix, you ask? Well, let's take it for a test drive and try out some values. First, let's try $s = 2$:

$$\zeta(2) = \frac{\pi^2}{6}.$$

Wait, we've seen that value before. That's $Z(2)$!

Now, let's do another one:

$$\zeta(4) = \frac{\pi^4}{90}.$$

That's $Z(4)$!

We can try value after value for s, but the same result will keep happening:

Riemann's Result: *For any s for which $Z(s)$ doesn't blow up, $\zeta(s) = Z(s)$.*

Riemann had found a function that mirrored $Z(s)$. Unlike $Z(s)$, though, $\zeta(s)$ didn't blow up if s was less than 1. And ζ was prepared to handle all kinds of numbers! Fractions! Decimals! Imaginary numbers like $\sqrt{-1}$! Combinations of real numbers and imaginary numbers! $\zeta(s)$ was like a post-spinach-Popeye version of $Z(s)$.

Unfortunately, like Achilles, ζ still had one flaw. There was one single value for s that ζ couldn't handle:

Fundamental Statement About Zeta: $\zeta(1)$ *is undefined.*

Oh well. You can't have everything.

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[3]Sometimes, in an attempt to get the same amount of shock value in less space, mathematicians will consolidate notation and write the function as

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi} \oint_{\gamma} \frac{u^{z-1}}{e^{-u} - 1} du.$$

For those who are interested, there's a more thorough explanation of how Riemann got from $Z(s)$ to $\zeta(s)$ in Appendix A.

3.1   Getting Back To Our Original Question

Now that we've found a suitable replacement for the mercurial $Z(s)$, we can try asking Euler's question again:

The Euler Question (again): *Is there anything interesting about $\zeta(s)$?*

This question can often be a dog-whistle type question for mathematicians; when we say, "Is it interesting?", we often mean, "Does it hit zero a lot?" That may not be interesting to everyone, but mathematicians think zero is really, really fascinating. We're kind of like that kid with the video camera in "American Beauty" who thinks that a bag blowing in the wind is the most beautiful thing in the world; you may think we're weird, but, well, who asked you anyway?

Since "interesting" means different things to different people anyway, let's try thinking about this zero stuff and see if it gets us anywhere:

The Euler Question (yet again): *When does $\zeta(s)$ hit zero?*

There are actually quite a few places where it is easy to show it hits zero:

Partial Answer: $\zeta(s) = 0$ *when* $s = -2, -4, -6, -8, -10, ...$

OK. We're partway there. Those are the ones that are actually pretty easy to find. What about other ones?

The Euler Question (for the last time, I swear): *Okay, okay. Besides negative even integers, when does $\zeta(s)$ hit zero?*

We don't have all the answers to this question, but here's one[4]:

$$\zeta(\frac{1}{2} + 14.134725142i) = 0.$$

Here's another one:

$$\zeta(\frac{1}{2} + 21.022039639i) = 0.$$

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[4]In case you've forgotten, $i = \sqrt{-1}$.

And another:

$$\zeta(\frac{1}{2} + 25.010857580i) = 0.$$

And yet another:

$$\zeta(\frac{1}{2} + 30.424876126i) = 0.$$

Starting to notice a pattern? They all seem to be $\frac{1}{2}$ plus some multiple of i.

It would be natural to ask whether this pattern will continue. It's so natural to ask, in fact, that Riemann beat you to it by over 150 years.

The Riemann Question: *Let's ignore those negative even integers for now. If $\zeta(s) = 0$, does that mean that s is $\frac{1}{2}$ plus a multiple of i?*

This question has a less than satisfactory answer:

The Riemann Answer: *I have no earthly idea.*

Coming up with a more satisfactory answer (such as "yes" or possibly even "no") is such a difficult thing to do that a $1,000,000 reward has been promised to the person who finally does. The best we currently have is the following:

The Riemann Guess: *The answer looks like it should be yes (?)*

or, in mathier speak:

The Riemann Hypothesis (The Riemann Guess with Fancier Words): *If $\zeta(s) = 0$ and s is not a negative even integer then s is $\frac{1}{2}$ plus a multiple of i.*

or, in even mathier speak:

The Riemann Hypothesis (Take 2): *If $\zeta(s) = 0$ and s is not a negative even integer than $s = \frac{1}{2} + it$ for some real number t.*

The Riemann Hypothesis is considered by many mathematicians to be the most important unsolved problem in mathematics today.

3.2   Wait, Wait, That's It? The Question of When Some Esoteric Function Hits Zero is The Most Important Problem In Math?

Yep.

3.3   How?

I'm glad you asked. Because the Riemann Zeta Function is based on such a simple equation (namely, $Z(s)$), it's something that comes up in a lot of computations, so having a good understanding of it would help us calculate all sorts of interesting properties about integers and prime numbers. As far as conjectures go, it's not as sexy (or as likely to generate crank mail) as something like the Twin Primes Conjecture, but we all know that beauty is skin-deep, and the Riemann Hypothesis is nothing if not deep[5].

3.4   Can You Give An Example of Something That the Riemann Hypothesis Can Show Us?

Funny you should ask. The next chapter is all about one of the most famous examples: Gauss' Prime Number Theorem.

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[5]Heck, it took me like ten pages to define the stupid thing. That's pretty deep.

Chapter 4

Application: Primes on Parade

4.1   Boredom is the Mother of Invention

Let us begin with a question which is actually three questions:

Question(s): *How many prime numbers are there up to 100? How about up to 1,000? Or 1,000,000?*

One way of answering the above is to manually count all of the primes up to 100 or 1,000 or 1,000,000. I would do that, except that it sounds like a lot of work and would be kind of boring, and, besides, the Red Sox game is on. So that's out.

   We are forced, then, to rephrase our question:

Better Question: *Counting is boring, and I'm lazy. Is there some formula that I could use where I just plug in some number and it will do all the work for me?*

This was the question that a fifteen-year old named Carl Friedrich Gauss considered in 1792. Unlike me, Gauss did not have an urgent Red Sox game to attend to, so he sat down, looked through the data, and came up with a pretty good answer:

Gauss' Answer: *The number of primes less than x is about $\frac{x}{\ln x}$.*

Gauss was not a baseball fan.

So if we wanted to know the number of primes less than 1,000, we could just calculate $\frac{1,000}{\ln 1,000}$ and we've got a pretty reasonable estimate.

This work should have been impressive enough for somebody who wasn't yet old enough to drive[1]. However, Gauss wasn't satisfied and announced, "No! I can do even better! I shall come up with a function that comes even closer to the correct number of primes! And the function will be easy to calculate!"

Although this sounded like a bunch of empty promises to the wary public in an election year like 1792, Gauss actually succeeded in his quest and found exactly the function he was looking for. He called this function Li because he, like most 18th century number theorists, was a big fan of kung-fu legend Bruce Lee[2].[3] Since Li actually turns out to be pretty straightforward to

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[1] Although I suppose this was less of an issue before the invention of the automobile.

[2] Oddly enough, Li is also the first letters of the words "Logarithmic Integral." Coincidence? Probably. I'm still going with the Bruce Lee explanation for the name.

[3] If you're curious, the actual definition of Li is given by

$$Li(x) = \int_2^x \frac{dt}{\ln t}.$$

calculate, if we want to find the number of primes up to 1,000, we can just calculate $Li(1,000)$ and we've got an even better estimate than before[4]. This realization deserves bold letters:

The Prime Number Theorem: *The number of primes less than x is approximately $Li(x)$.*

4.2 Riemann Hypothesis: Karate Kicking Li's Error Terms Since 1859

In the last section, we made a lot of nebulous statements like, "This is a good estimate," or "This is an even better estimate." This is unfortunate, as mathematicians don't like ambiguously descriptive words like "good" because they don't really tell us anything. We are forced, then, to ask the question, "How good of an estimate is this, anyway?" Annoyingly, this question also uses the word "good," but we'll let it slide because it's a useful question; in fact, it is such an important question that it deserves italics:

The Li Question: *How closely does Li estimate the number of primes less than x?*

As before, we start by naming things. The most obvious candidate for a makeover is "the number of primes less than x," which is annoying to write out all the time. To help with this, I'm going to call this quantity $P(x)$.[5] Armed with this notation, we rephrase the question and pretend that we've accomplished something in doing so:

If you weren't curious, well, too bad. It's your fault for looking down here at the footnote in the first place.

[4]It should be noted that while Gauss came up with these guesses, they weren't proven to be correct until 1896. In other words, instead of taking the five minutes to count the number of primes up to 1,000, mathematicians spent hours coming up with a guess for what the answer should be, then spent over a hundred years proving that the guess was correct. Yep, that was productive.

[5]Mathematicians actually call this $\pi(x)$; however, since we already have $\pi = 3.14159....$, I didn't want to confuse the readership.
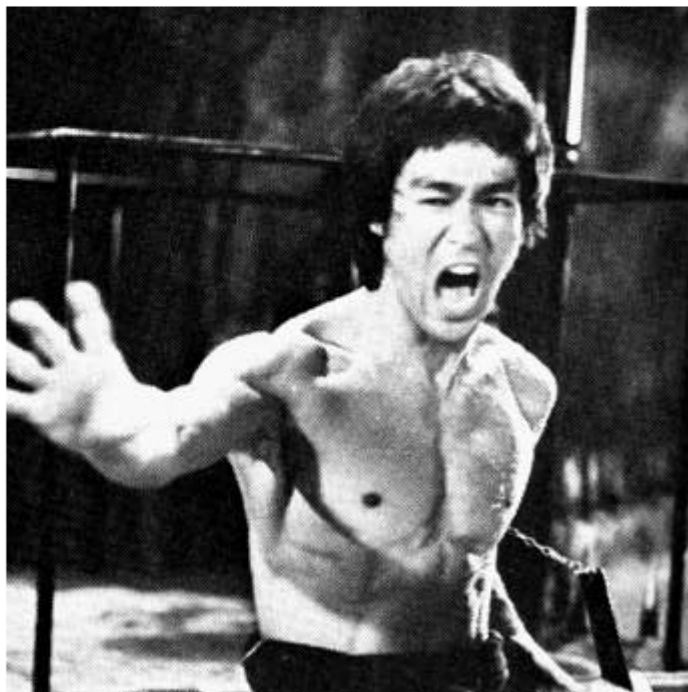
A reasonable approximation for P(x).

The More Economical Li **Question**: *How far apart do $Li(x)$ and $P(x)$ get?*

Sometimes, we call the difference between $Li(x)$ and $P(x)$ the *error term in the Prime Number Theorem*. Sometimes, we don't. Really, it just depends how we're feeling.

In our pre-Riemann Hypothesis world, the answer to the above question was unsatisfying:

The Li **"Answer"**: *Well, we think so.*

In fact, we know that $Li(x)$ and $P(x)$ can differ by as much as $\sqrt{x} \cdot ln\ x$ because we've seen it in actual data. The hope is that they don't differ by much more than that because $\sqrt{x} \cdot ln\ x$ is pretty small relative to the number of things we're counting, which would mean that the Li function does a really good job of approximating $P(x)$. In the last hundred years, mathematicians have come up with better and better answers, but we're still nowhere near

where the data indicates that we should be.

On the other hand, if the Riemann Hypothesis were true, we would have a very, very good answer to this question:

The *Li* Answer with Riemann: *If the Riemann Hypothesis is true then $Li(x)$ and $P(x)$ never differ by more than about $\sqrt{x} \cdot \ln x$.*

In other words, what we hope to be true is actually true if the Riemann Hypothesis is correct. That's a pretty powerful hypothesis.

Our story would normally end here, except that there's actually a weird sidenote to this:

The *Li* Answer with Riemann, but Backwards: *If $Li(x)$ and $P(x)$ never differ by more than about $\sqrt{x} \cdot \ln x$ then the Riemann Hypothesis is true.*

So as it turns out, the question about the error term and the Riemann Hypothesis are *actually the same question*; if you prove one of them, the other is necessarily true.

Man, math is weird sometimes.

Chapter 5

Appendix A: Analytic Continuation

In this section, I'll discuss a little more of the math that goes on. If you aren't really interested in the math behind it, then.....wait, seriously? You spent the last twenty pages reading about the Riemann Hypothesis but don't care about math? What kind of garbage is that? Suck it up and keep reading.

5.1 Riemann's Number Theoretical Patch: Better than Number Theoretical Gum

Let's go back to Riemann's idea. When we last left our hero, he had stumbled upon the following creature:

$$Z(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + ...$$

This function was having a hard time getting through the world because it exploded at the mere mention of numbers less than or equal to 1. Riemann felt that this function would benefit from some sort of patch, so he decided to try multiplying it by $(1 - \frac{2}{2^s})$.[1] Armed with this patch, Z became a little less temperamental:

$$(1 - \frac{2}{2^s})Z(s) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s}....$$

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[1]This is quite similar to the movie "Patch Adams", wherein Patch fixes his friend's leaky cup by multiplying it by $(1 - \frac{2}{2^s})$.

The right hand side of this function is now defined when $s > 0$. That's progress.

How does this help? Well, Riemann decided to divide both sides of the above by $(1 - \frac{2}{2^s})$. This gives

$$Z(s) = \frac{\frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} \dots}{1 - \frac{2}{2^s}}.$$

Now, he had an expression for Z that was defined for s's all the way down to zero. He decided to celebrate in traditional mathematics fashion by renaming his function[2]; from now on, this Z would become known as......ζ:[3]

$$\zeta(s) = \frac{\frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} \dots}{1 - \frac{2}{2^s}}.$$

5.1.1 An Obvious Symmetry

All right, so we've gotten s down to 0 instead of 1. How do we get s to go the rest of the way?

Well, as it turns out, if s is between 0 and 1, $\zeta(s)$ has a very nice symmetry to it. In fact, I'm sure you noticed this symmetry immediately, so I don't need to point it out, but I'll do so anyway:

$$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{s\pi}{2}) \left[\int_0^\infty e^{-y} y^{-s} dy \right] \zeta(1-s)$$

when s is between 0 and 1. That was obvious.

Now, note that if you plugged in, say, $\frac{1}{4}$ for s, you end up with $\zeta(\frac{3}{4})$ on the right-hand side. This isn't much of an improvement, because we can already calculate $\zeta(\frac{1}{4})$, so we basically just took something we knew and made it harder.

But Riemann had another thought. What if you plugged in, say, -3 for s? Then the above becomes:

$$\zeta(-3) = 2^{-3} \pi^{-4} \sin(\frac{-3\pi}{2}) \left[\int_0^\infty e^{-y} y^3 dy \right] \zeta(4).$$

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[2]We mathematicians know how to throw rollicking parties.

[3]Note that ζ is still not defined for $s = 1$ because if we plugged in 1 for s, the bottom would be zero and division by zero is bad. Very, very bad.

This is ugly. However, it's doable. Or rather, the right side is doable; $\zeta(4)$ can be easily calculated, and there's no reason you couldn't calculate the rest of the things on the right hand side as well[4]. But wait - we've got an equal sign, and the thing on the right is defined - the thing on the left is defined, too (and equals the thing on the right). So $\zeta(-3)$ is finally defined! And this definition doesn't even involve the word "blow".

What's special about -3? Nothing! We could have done this for any negative number. In other words, if I wanted to find $\zeta(s)$ for some negative value of s, I could simply plug this negative value into the expression above; the right-hand side would give me a bunch of stuff I could calculate, including ζ of some positive number (which has already been defined). Armed with this trick, we can now evaluate ζ for anything - except, of course, $s = 1$.[5]

And thus, the Z-function was made whole.

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[4]Note that I say that *you* can calculate it because I certainly have no interest in doing so.

[5]In case you're wondering, mathematicians have a word for a function like this that works nicely at every possible s except for some limited number of values (in this case, one); that word is *meromorphic*. They also have some words for the points where the function isn't defined, but those words are a bit too colorful for this book and need not be repeated here.